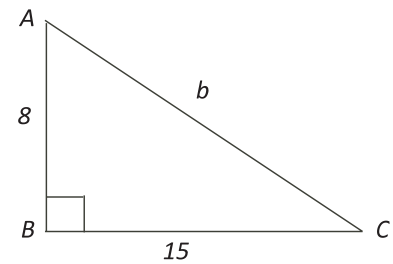
Mathematics Teachers Enrichment Program
MTEP 2012
Trigonometry and Bearings
Solutions to Exercises

1. In $\triangle ABC$, $A\hat{B}C = 90^\circ$, $AB = 8$ and $BC = 15$. Solve $\triangle ABC$. Round side length and angles to one decimal, as necessary.



Solution:

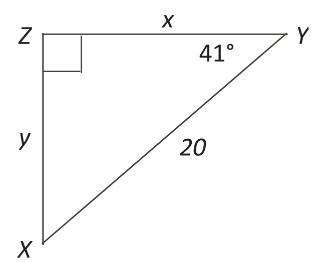
Let b represent the length of side AC.

Using Pythagoras' Theorem, $b^2 = 8^2 + 15^2 = 289$ and $b = 17$ follows.

Using basic trigonometry, $\tan A\hat{C}B = \tan \hat{C} = \frac{8}{15} \doteq 0.5333$ and $\hat{C} = 28.1^\circ$.
Since the angles in a triangle add to 180°, $B\hat{A}C = \hat{A} = 180^\circ - 90^\circ - 28.1^\circ = 61.9^\circ$.

Therefore $A\hat{C}B = 28.1^\circ$, $B\hat{A}C = 61.9^\circ$ and $|AC| = 17$.

2. Solve $\triangle XYZ$, given $X\hat{Z}Y = 90^\circ$, $X\hat{Y}Z = 41^\circ$, and $XY = 20$. Round side lengths to one decimal, as necessary.



Solution:

Let x represent the length of side ZY and y represent the length of side XZ.
Since the angles in a triangle add to 180°,
$Z\hat{X}Y = \hat{X} = 180^\circ - 90^\circ - 41^\circ = 49^\circ$.
Using basic trigonometry, $\sin 41^\circ = \frac{y}{20}$ and
$y = 20 \sin 41^\circ \doteq 13.1$ follows.

Using Pythagoras' Theorem, $x^2 = 20^2 - 13.12^2 = 227.83$. $x \doteq 15.1$ follows. You could also use basic trigonometry here again. $\cos 41^\circ = \frac{x}{20}$ and $x = 20 \cos 41^\circ \doteq 15.1$ follows.

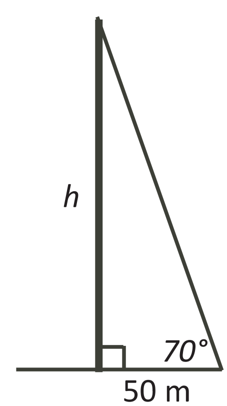
Therefore $Z\hat{X}Y = 49^\circ$, $|ZY| = 15.1$ and $|XZ| = 13.1$.

3. At a point 50 m from the base of a tower the angle of elevation to the top of the tower is 70°. Determine the height of the tower, to the nearest metre.

Solution:

Let h represent the height of the tower, in m.

$$\begin{aligned} \text{Using basic trigonometry, } \tan 70^\circ &= \frac{h}{50} \\ h &= 50 \tan 70^\circ \\ h &\doteq 137.4 \end{aligned}$$

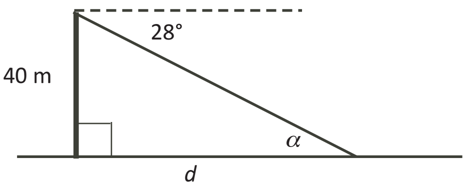


Therefore, to the nearest metre, the height of the tower is 137 m.

4. From the top of a 40 m tower, an observer measures the angle of depression to a boat in the water below to be 28°. How far, to the nearest metre, is the boat from the base of the tower?

Solution:

Let d represent the distance, in m, from the bottom of the tower to the boat. Let α represent the angle of elevation from the boat to the top of the tower. Therefore $\alpha = 28°$. (The angle of elevation from the boat to the top of the tower equals the angle of depression from the top of the tower to the boat.)



Using basic trigonometry,

$$\tan 28° = \frac{40}{d} \text{and } d = \frac{40}{\tan 28°} \doteq 75.2 \text{ m follows.}$$

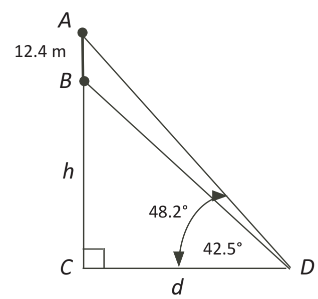
Therefore, to the nearest metre, the boat is 75 m out from the bottom of the tower.

5. A 12.4 m flagpole is placed on top of a tall building. An observer, standing directly in front of the building and flagpole, measures the angle of elevation to the bottom of the flagpole to be 42.5° and to the top of the flagpole to be 48.2°. Determine the height of the building, to the nearest metre.

Solution:

Represent the given information on a diagram. Let the height of the building be h and the distance out from the building to the observer be d.

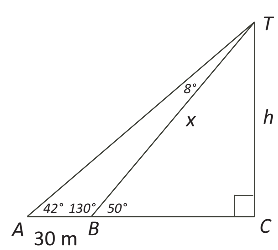
In $\triangle BCD$, $\frac{h}{d} = \tan 42.5°$ and in $\triangle ACD$, $\frac{h+12.4}{d} = \tan 48.2°$. Rearranging, $h = d(\tan 42.5°)$ and $h + 12.4 = d(\tan 48.2°)$. Substitute for h in the second equation,



$$\begin{aligned} d(\tan 42.5°) + 12.4 &= d(\tan 48.2°) \\ 12.4 &= d(\tan 48.2°) - d(\tan 42.5°) \\ 12.4 &= d(\tan 48.2° - \tan 42.5°) \\ 12.4 \div (\tan 48.2° - \tan 42.5°) &= d \\ 61.35 &\doteq d \\ \text{But } h &= d(\tan 42.5°) \doteq 56.2 \end{aligned}$$

Therefore the height of the building is 56 m.

6. To calculate the height of a tower, David measured the angle of elevation of the top of the tower from a point A to be 42°. He then moved 30 m closer to the tower and from point B measured the angle of elevation to the top of the tower to be 50°. To the nearest metre, determine the height of the tower.



Solution:

$A\hat{B}T$ and $T\hat{B}C$ form a straight angle. Therefore $A\hat{B}T = 180° - 50° = 130°$.

The angles in a triangle add to 180° so in $\triangle TBA$, $A\hat{T}B = 180° - 42° - 130° = 8°$.

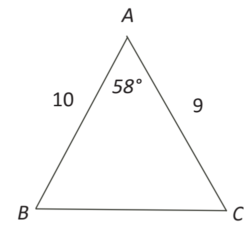
Let x represent the length of side BT and h represent TC, the required height.

Using the Sine Rule in $\triangle ABT$, $\frac{x}{\sin 42°} = \frac{30}{\sin 8°}$ and $x = \frac{30 \sin 42°}{\sin 8°} \doteq 144.24$.

Then in $\triangle TBC$, $\frac{h}{x} = \sin 50°$ and $h = x \sin 50 \doteq 110$

The height of the tower is 110 m.

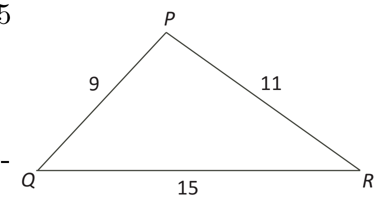
7. Determine the length of BC in $\triangle ABC$ where $AB = 10$, $AC = 9$ and $B\hat{A}C = 58°$. Round correctly to one decimal place.



Solution:

The diagram to represent the problem is shown to the right. Using the Cosine Rule, $|BC|^2 = 10^2 + 9^2 - 2 \times 10 \times 9 \cos 58° \doteq 85.61$ and $|BC| \doteq 9.3$.

8. Determine the size of the largest angle in $\triangle PQR$ where $PQ = 9$, $QR = 15$ and $PR = 11$. Round correctly to one decimal place.

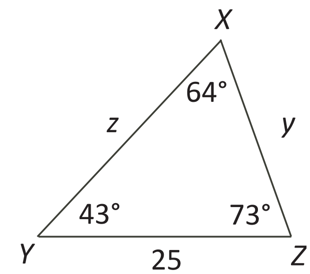


Solution:

The largest angle will be opposite the largest side. Therefore we are required to find $\hat{P}$.

Using the Cosine Rule, $\cos \hat{P} = \dfrac{9^2 + 11^2 - 15^2}{2(9)(11)} \doteq -0.1162$ and $\hat{P} \doteq 96.7°$.

9. In $\triangle XYZ$, $X\hat{Y}Z = 43°$, $X\hat{Z}Y = 73°$ and $YZ = 25$. Solve $\triangle XYZ$, rounding correctly to one decimal place.



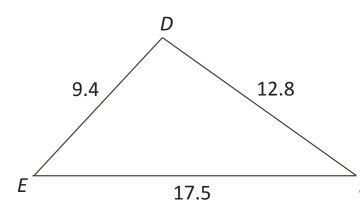
Solution:

Since two angles in the triangle are known, the third angle is easily found. $\hat{X} = 180° - 43° - 73° = 64°$.

Using the Sine Rule, $\dfrac{z}{\sin 73°} = \dfrac{y}{\sin 43°} = \dfrac{25}{\sin 64°}$.

It follows that $z = \dfrac{25 \sin 73°}{\sin 64°} \doteq 26.6$ and $y = \dfrac{25 \sin 43°}{\sin 64°} \doteq 19.0$.

10. In $\triangle DEF$, $DE = 9.4$, $EF = 17.5$ and $DF = 12.8$. Solve $\triangle DEF$. rounding each angle to one decimal place.



Solution:

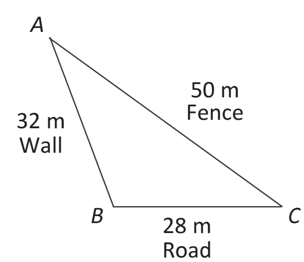
The diagram to represent the information from the problem is shown to the right. Using the Cosine Rule, $\cos \hat{D} = \dfrac{12.8^2 + 9.4^2 - 17.5^2}{2 \times 12.8 \times 9.4} \doteq -0.2246$ and $\hat{D} \doteq 103.0°$.

Using the Sine Rule, $\dfrac{\sin \hat{F}}{9.4} = \dfrac{\sin 103.0°}{17.5}$. Then $\sin \hat{F} = \dfrac{9.4 \sin 103.0°}{17.5} \doteq 0.5234$ and $\hat{F} \doteq 31.6°$.

It follows that $\hat{E} = 180° - 103.0° - 31.6° = 45.4°$.

Therefore, $\hat{D} = 103.0°$, $\hat{E} = 45.4°$, and $\hat{F} = 31.6°$.

11. A triangular piece of land is bounded by 32 m of brick wall, 50 m of fencing and 28 m of road along the front. What angle does the fence make with the road?



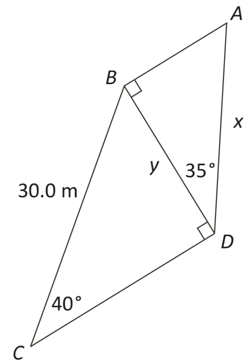
Solution:

Represent the given information on the diagram. We are looking for $\hat{C}$.

Using the Cosine Rule, $\cos \hat{C} = \dfrac{50^2 + 28^2 - 32^2}{2 \times 50 \times 28} \doteq 0.8071$ and $\hat{C} \doteq 36.2°$.

Therefore, the fence meets the road at a 36° angle.

12. Determine the length of x to one decimal place.



Solution:

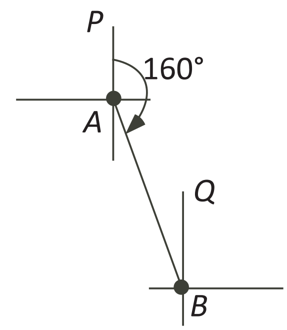
Let y represent the length of BD.

In $\triangle CBD$, $\dfrac{y}{30} = \sin 40°$, $y = 30 \sin 40° \doteq 19.3$.

In $\triangle ABD$, $\dfrac{y}{x} = \cos 35°$, $x = y \div \cos 35° \doteq 23.5$.

Therefore the length of x is 23.5 units.

13. The bearing from A to B is 160°. What is the bearing from B to A?

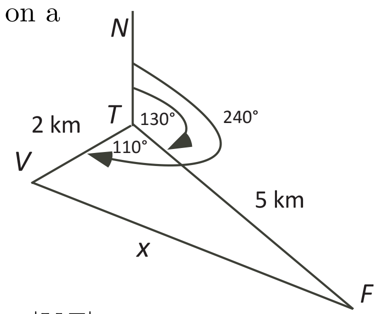


Solution:

Represent the given information on the diagram. $P\hat{A}B = 160°$ represents the bearing from A to B. We want the bearing from B to A and therefore want to find reflex angle $A\hat{B}Q$.

$PA \parallel QB$ so $A\hat{B}Q = 180° - P\hat{A}B = 180° - 160° = 20°$. Reflex angle $A\hat{B}Q = 360° - 20° = 340°$. Therefore the bearing from B to A is 340°.

14. From an observation point in a fire tower, the observer spots a fire 5 km away at a bearing 130°. The observer also spots a village 2 km away on a bearing 240°. How far is the fire away from the village?



Solution:

Represent the given information on the diagram.

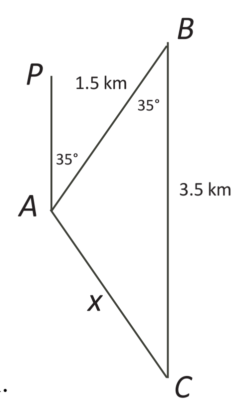
$N\hat{T}F = 130°$ is the bearing from the tower to the fire.
$N\hat{T}V = 240°$ is the bearing from tower to the village.
Then $V\hat{T}F = 240° - 130° = 110°$.

Let x represent the distance from the village to the fire. Therefore $x = |VF|$.

By the Cosine Rule, $x^2 = 2^2 + 5^2 - 2(2)(5)\cos 110° \doteq 35.84$ and $x \doteq 6.0$ km.

The distance from the fire to the village is 6.0 km.

15. A hiker walks 1.5 km on a bearing 035°. At this point he turns directly south and walks 3.5 km. How far and on what bearing must he walk to return to his original starting point?



Solution:

Represent the given information on the diagram.

$P\hat{A}B = 35°$ is the bearing for the 1.5 km part of the hike, shown as A to B. Then the hiker turns south and goes 3.5 km to C. Therefore $PA \parallel BC$ and $A\hat{B}C = P\hat{A}B = 35°$ follows.

Let x represent the distance from C to the original start point at A.

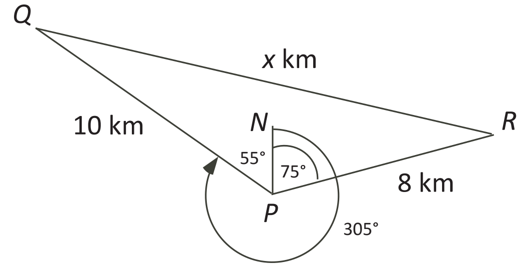
By the Cosine Rule, $x^2 = 1.5^2 + 3.5^2 - 2(1.5)(3.5)\cos 35° \doteq 5.90$ and $x \doteq 2.4$ km.

Using the Sine Rule, $\dfrac{\sin \hat{C}}{1.5} = \dfrac{\sin 35°}{x}$. Then $\sin \hat{C} = \dfrac{1.5 \sin 35°}{x} \doteq 0.3542$ and $\hat{C} \doteq 20.7°$.

The required bearing is $360° - 20.7° \doteq 339.3°$.

The hiker must walk about 2.4 km on a bearing of 339° to get to the original start point.

16. Two ships, R and Q, left a port, P, at the same time. After two hours, R had travelled 8 km on a bearing 075° from port and Q had travelled 10 km on a bearing 305° from port. How far apart are the two ships after two hours?



Solution:

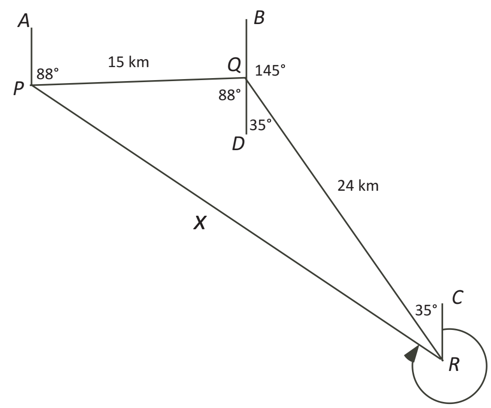
Let x represent the distance between the two boats after 2 hours. Represent the given information on the diagram.

$N\hat{P}R = 75°$ is the bearing from P to R. Reflex angle $N\hat{P}Q = 305°$ is the bearing from P to Q. Then $N\hat{P}Q = 360° - 305° = 55°$ and $Q\hat{P}R = 55° + 75° = 130°$.

By the Cosine Rule, $x^2 = 10^2 + 8^2 - 2(10)(8)\cos 130° \doteq 266.85$ and $x \doteq 16.3$ km.

The ships are 16.3 km apart after two hours.

17. A tour boat leaves port and travels 15 km on a bearing of 088° and then travels a further 24 km on a bearing of 145°. The boat then returns directly to the starting point. Determine the distance to the port and the bearing along which the tour boat must travel.



Solution:

Let x represent the distance the tour boat must travel to return to port. Represent the given information on the diagram. Other information on the diagram will be justified below.

$A\hat{P}Q = 88°$ is the bearing from the port for 15 km. $AP \parallel BD$ so $P\hat{Q}D = A\hat{P}Q = 88°$.

$B\hat{Q}R = 145°$ is the bearing for the 24 km part of the tour. BQD is a straight line so $D\hat{Q}R = 180° - 145° = 35°$. $CR \parallel BD$ so $Q\hat{R}C = D\hat{Q}R = 35°$.

$P\hat{Q}R = P\hat{Q}D + D\hat{Q}R = 88° + 35° = 123°$.

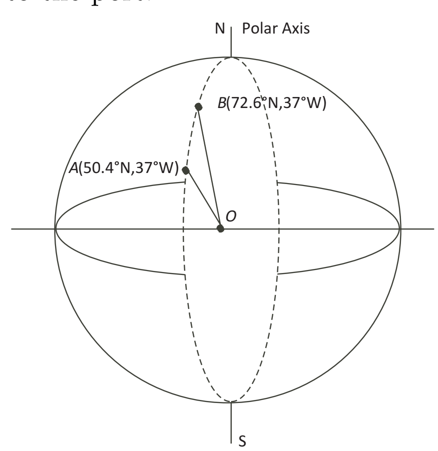
Using the Cosine Rule, $x^2 = 15^2 + 24^2 - 2(15)(24)\cos 123° \doteq 1193.14$ and $x \doteq 34.5$.

Using the Sine Rule, $\dfrac{\sin P\hat{R}Q}{15} = \dfrac{\sin 123°}{x}$ and $\sin P\hat{R}Q = \dfrac{15 \sin 123°}{x} \doteq 0.3642$. Then $P\hat{R}Q \doteq 21°$.

The required bearing is the reflex angle $C\hat{R}P = 360° - P\hat{R}Q - Q\hat{R}C \doteq 360° - 21° - 35° = 304°$.

The tour boat must travel 34.5 km on a bearing of 304° to return to the port.

18. Two towns, A and B, lie on longitude 37°W. Their latitudes are 50.4°N and 72.6°N, respectively. Calculate the shortest distance between the two towns.



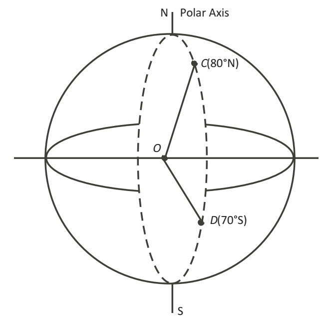
Solution:

The sector angle is $\theta = 72.6° - 50.4° = 22.2°$.

The distance from A to B along longitude 37°W is

$\dfrac{\theta}{360} \times 2\pi r = \dfrac{22.2}{360} \times 40\,000 \doteq 2\,470$ km, to three significant figures.

19. Two towns, C and D, lie on the same longitude. Their latitudes are 80°N and 70°S, respectively. Calculate the shortest distance between the two towns.



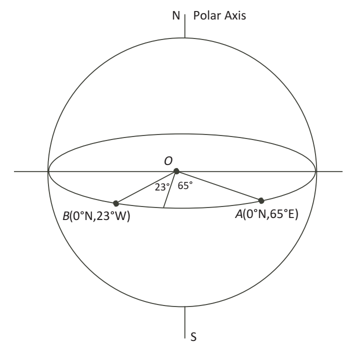
Solution:

The sector angle is found by adding the two latitudes $\theta = 80° + 70° = 150°$.

The distance from C to D along the same longitude is

$\frac{\theta}{360} \times 2\pi r = \frac{150}{360} \times 40\,000 \doteq 16\,700$ km, to three significant figures.

20. Two cities, A and B, lie on the Equator, one at longitude 65°E and the other at longitude 23°W. Calculate the distance between the two towns, measured along the Equator.



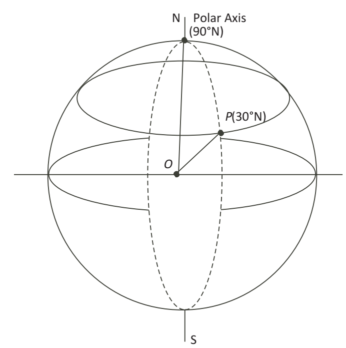
Solution:

The sector angle is found by adding the two longitudes $\theta = 65° + 23° = 88°$.

The distance from A to B along the Equator is

$\frac{\theta}{360} \times 2\pi r = \frac{88}{360} \times 40\,000 \doteq 9\,780$ km, to three significant figures.

21. Find the distance, measured along a meridian, from any point on parallel of latitude 30°N to the North Pole.



Solution:

The sector angle is found by subtracting the latitude from 90° giving 60°.

The distance measured along any meridian is

$\frac{\theta}{360} \times 2\pi r = \frac{60}{360} \times 40\,000 \doteq 6\,670$ km, to three significant figures.

22. Two towns are located on longitude 50°E. The towns are 5 200 km apart. Calculate the difference in their latitudes.

Solution:

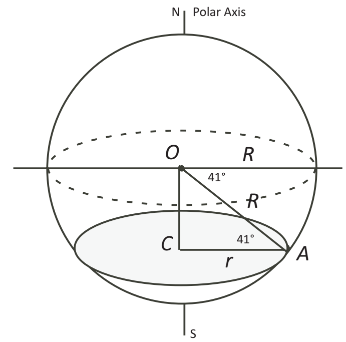
We are looking for θ in the formula $d = \frac{\theta}{360°} \times 2\pi R$. $d = 5200$ km and $2\pi R \doteq 40\ 000$, the circumference of the Earth.

$$\begin{aligned} 5\,200 &= \frac{\theta}{360} \times 40\,000 \\ \theta &= 5\,200 \times 360 \div 40\,000 \\ \theta &\doteq 47° \end{aligned}$$

The two towns are 47° apart on longitude 50°E.

23. Find the length of the parallel of latitude 41°S.

Solution:



We know that $r = R\cos\theta$ where r is the radius of the small circle on latitude θ (N or S), and R is the radius of the Earth. So $r = R\cos 41°$.

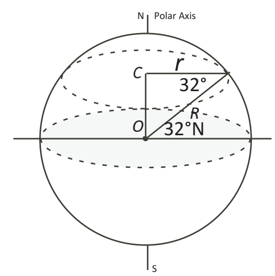
The length of the small circle known as latitude 41°S is $2\pi r = 2\pi R\cos 41° = 40\,000\cos 41° \doteq 30\,200$ km.

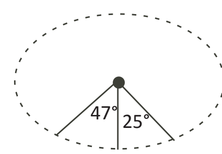
24. Two places are located on the parallel of latitude 32°N, one at longitude 47°W and the other at longitude 25°E.

a) How far apart are the two towns, measured along the parallel of latitude 32°N?

b) If it takes a plane 10 hours to travel from one town to the other, calculate the speed of the plane, correct to the nearest kilometer per hour.

Solution:





a) We know that $r = R\cos\theta$ where r is the radius of the small circle on latitude θ (N or S), and R is the radius of the Earth. So $r = R\cos 32°$.

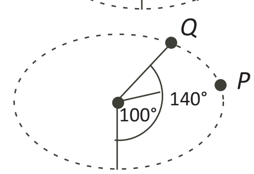
The angle between the places on latitude 32°N is $47° + 25° = 72°$.

The distance between the two places is

$$\frac{72°}{360°} \times 2\pi r = \frac{72}{360} \times 2\pi R\cos 32° = \frac{72}{360} \times 40\,000\cos 32° \doteq 6\,780 \text{ km}.$$

b) Using *speed* = *distance* ÷ *time*, speed $= 6780 \div 10 \doteq 678$ km/h.

The two places are 6 780 km apart and a plane travelling 678 km/h would take about 10 hours to get there.



25. Two towns, P and Q, are located at $(4°\text{N}, 100°\text{E})$ and $(4°\text{N}, 140°\text{E})$, respectively. Calculate the shortest distance from P to Q, along the parallel of latitude 4°N.

Solution:

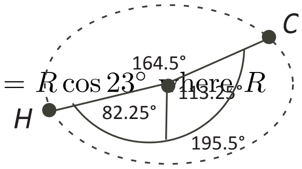
Both towns are located on latitude 4°N. The radius of latitude 4°N is $r = R\cos 4°$ where R is the radius of the Earth.

The angle between the places on latitude 4°N is $140° - 100° = 40°$.

The distance between the two places is $\frac{40°}{360°} \times 2\pi r = \frac{40}{360} \times 2\pi R\cos 4° = \frac{1}{9} \times 40\,000\cos 4° \doteq 4\,430$ km.

26. Havana and Canton are both on latitude 23°N, and their longitudes are 82.25°W and 113.25°E, respectively. Find their distance apart measured along the parallel of latitude.

Solution:



Both towns are located on latitude 23°N. The radius of latitude 23°N is $r = R\cos 23°$ where R is the radius of the Earth.

The angle between the places on latitude 23°N is $82.25° + 113.25° = 195.5°$. This angle is a reflex angle so we can go the other way $360° - 195.5° = 164.5°$ and travel a shorter distance.

The distance between the two places is

$$\frac{164.5°}{360°} \times 2\pi r = \frac{164.5}{360} \times 2\pi R\cos 23° = \frac{164.5}{360} \times 40\,000 \cos 23° \doteq 16\,800 \text{ km}.$$

The distance from Havana to Canton along latitude 23°N is 16 800 km to three significant figures.

27. Cherepovets, Russia is located approximately at $(60°\text{N}, 38°\text{E})$ and Mt. Logan, Canada is located approximately at $(60°\text{N}, 142°\text{W})$.

a) How far apart are the two locations, measured along the parallel of latitude 60°N?

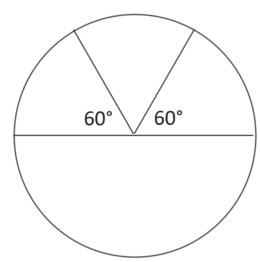
b) Calculate the great circle distance between Cherepovets and Mt. Logan.

Solution:

a) Both Cherepovets and Mt. Logan are on latitude 60°N and they are $38° + 142° = 180°$ apart. This means that on latitude 60°N they are one half the circumference of this small circle apart. We know that $r = R\cos 60°$ and hence the distance along the parallel of latitude 60°N is

$$\frac{1}{2} \times 2\pi r = 0.5 \times 2\pi R\cos 60° = 0.5 \times 40\,000 \times \cos 60° = 10\,000 \text{ km}.$$

b) But since their longitudes are 180° apart, both Cherepovets and Mt. Logan are located on a great circle through the North and South Poles. The sector angle is $180° - 2 \times 60° = 60°$.



The great circle distance between Cherepovets and Mt. Logan is

$$\frac{60°}{360°} \times 2\pi R = \frac{1}{6} \times 40\,000 \doteq 6\,670 \text{ km}.$$

Therefore Cherepovets and Mt. Logan are 10 000 km apart along latitude 60°N and 6 670 km apart along the great circle that passes through the North and South Poles.