Mathematics Teachers Enrichment Program
MTEP 2012
Sets and Forms

Definitions and Terminology

Sets

A *set* is a collection of distinct (different) objects.

Elements

The individual objects contained in a set are called *elements* or *members*.

Set Notation

When the elements of a set **can** be listed, they are written inside brace brackets, $\{\ \ \}$, separated by commas. The order in which the elements are listed is not important. However, sometimes there may be an advantage to listing the elements in a particular order. It is customary to name a set using an uppercase letter of the alphabet.

Size of a Set

We are generally interested in counting the number of elements in a set. Sometimes we may not be able to list a set but we are still able to count the number of elements in it. For a set A, the number of elements in the set is written $n(A)$.

If $A = \{\text{red, blue, green, yellow}\}$ then $n(A) = 4$ since the set contains four elements.

Examples:

For each given description, list the set and determine the number of elements in the set.

1. A : the days of the week
   Solution:

   $$A = \{\text{Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}\}$$

   $$\text{n}(A) = 7$$

2. B : the positive odd numbers less than or equal to 20
   Solution:

   $$B = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$$

   $$\text{n}(B) = 10$$

3. C : the people in this room
   Solution:

$$C = \{\text{Al, Bo, Ca, Dan, Ed, Fin, Gil, Hal, Ida, Jin, Kip, Lin, Mai, Ned, Olga, Pat, Quin, Ray, Sue, Tina, Ula, Vana, Wes, Xena, Yei, Zea}\}$$

$$\text{n}(C) = 26$$

4. D : the cards in a standard deck of playing cards
   Solution:

$$\begin{aligned} D = \{&1\clubsuit, 2\clubsuit, 3\clubsuit, 4\clubsuit, 5\clubsuit, 6\clubsuit, 7\clubsuit, 8\clubsuit, 9\clubsuit, 10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit,\\ &1\diamondsuit, 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit,\\ &1\heartsuit, 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit,\\ &1\spadesuit, 2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit\}\end{aligned}$$

$$\text{n}(D) = 52$$

More Terminology

Universal Set
The *Universal Set*, U, is the set containing all elements under consideration in a particular situation. Each of the sets listed in our above examples will be used as universal sets later.

Empty or Null Set
A set containing no elements is called the *empty set* or *null set*. If set P contains no elements we would denote it $P = \varnothing$ or $P = \{\ \}$.

Subsets
If all the elements of set X are also elements of set Y, we say that set X is a *subset* of set Y and write $X \subseteq Y$. We say "X is contained in Y" or "Y contains X". According to this definition, every set is a subset of itself and the empty set is a subset of every set.

Disjoint Sets
When two sets have no elements in common, the sets are said to be *disjoint*.

Equal Sets and Equivalent Sets
Sets that contain exactly the same elements are called *equal* sets. Sets that contain the same number of elements are called *equivalent* sets.

Complement of a Set
The *complement* of a set A is a subset of the universal set U containing all the elements from U that are not in A. We denote the complement of A as A'. It should also be noted that $\text{n}(U)=\text{n}(A)+\text{n}(A')$. (It could also be noted that complements are disjoint sets.)

Venn Diagrams
Relationships between sets and their subsets can be illustrated using *Venn diagrams*. In such a diagram we use a rectangle to represent the universal set and closed curves, usually circles, to represent the sets in which we are interested.

Examples:

For each of the following, list the sets in set notation. Illustrate the sets on a Venn diagram. The Universal set is listed for you. Classify the subsets as equal, equivalent, disjoint or complements.

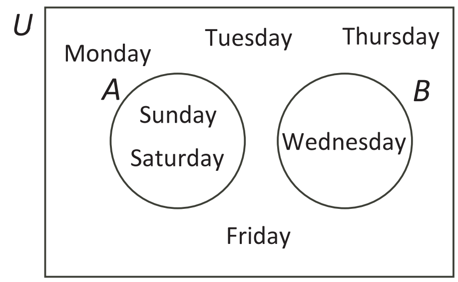
5. $U = \{\text{Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}\}$
   A : the days of the week starting with the letter S
   B : the days of the week containing 9 letters

   Solution:

   $A = \{\text{Sunday, Saturday}\}$

   $B = \{\text{Wednesday}\}$

   A and B are disjoint subsets of U.



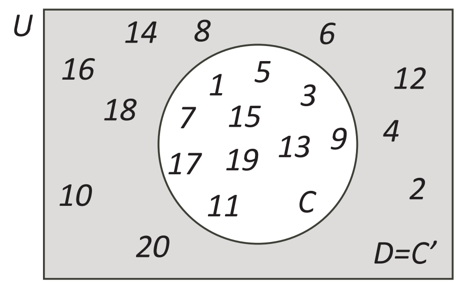
6. $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$
   C : the odd integers in U
   D : the even integers in U

   Solution:

   $C = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

   $D = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$

   C and D are complements. Also C and D are equivalent since they have the same number of elements, $n(C) = n(D) = 10$. C and D are also disjoint subsets.



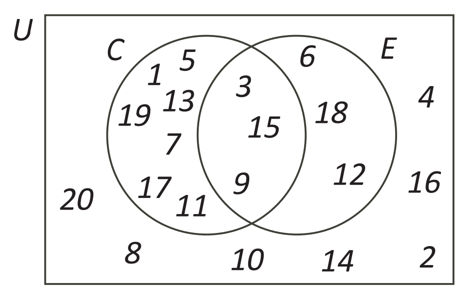
7. $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$
   C : the odd integers in U
   E : multiples of 3 in U

   Solution:

   $C = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

   $D = \{3, 6, 9, 12, 15, 18\}$

   C and E are not complements nor are they disjoint nor are they equal nor are they equivalent. There are elements in C that are in E and elements in E that are also in C.



Example 8

Using the cards in a standard deck of playing cards, create subsets of U that are (i) disjoint, (ii) complements, (iii) empty, (iv) equal, and (v) equivalent. Write the sets using set notation and illustrate the sets on a Venn diagram.

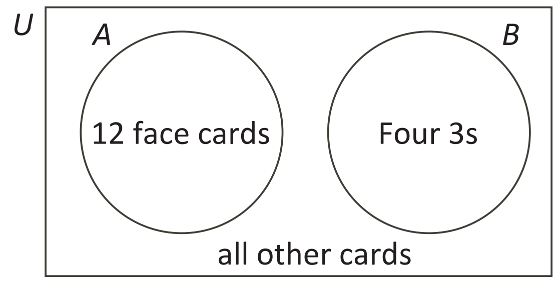
$$\begin{aligned} U = \{&1\clubsuit,\ 2\clubsuit,\ 3\clubsuit,\ 4\clubsuit,\ 5\clubsuit,\ 6\clubsuit,\ 7\clubsuit,\ 8\clubsuit,\ 9\clubsuit,\ 10\clubsuit,\ J\clubsuit,\ Q\clubsuit,\ K\clubsuit,\\ &1\diamondsuit,\ 2\diamondsuit,\ 3\diamondsuit,\ 4\diamondsuit,\ 5\diamondsuit,\ 6\diamondsuit,\ 7\diamondsuit,\ 8\diamondsuit,\ 9\diamondsuit,\ 10\diamondsuit,\ J\diamondsuit,\ Q\diamondsuit,\ K\diamondsuit,\\ &1\heartsuit,\ 2\heartsuit,\ 3\heartsuit,\ 4\heartsuit,\ 5\heartsuit,\ 6\heartsuit,\ 7\heartsuit,\ 8\heartsuit,\ 9\heartsuit,\ 10\heartsuit,\ J\heartsuit,\ Q\heartsuit,\ K\heartsuit,\\ &1\spadesuit,\ 2\spadesuit,\ 3\spadesuit,\ 4\spadesuit,\ 5\spadesuit,\ 6\spadesuit,\ 7\spadesuit,\ 8\spadesuit,\ 9\spadesuit,\ 10\spadesuit,\ J\spadesuit,\ Q\spadesuit,\ K\spadesuit\} \end{aligned}$$

Solution:

(i) Disjoint subsets

A: face cards in U
B: threes in U

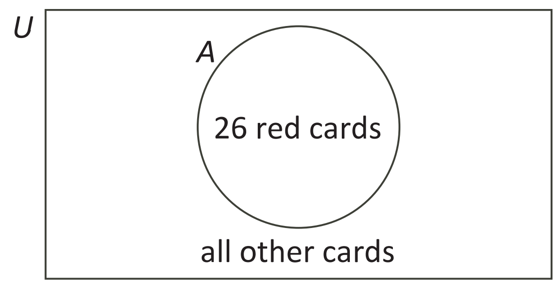
$A=\{J\clubsuit,\ Q\clubsuit,\ K\clubsuit,\ J\diamondsuit,\ Q\diamondsuit,\ K\diamondsuit,\ J\heartsuit,\ Q\heartsuit,\ K\heartsuit,\ J\spadesuit,\ Q\spadesuit,\ K\spadesuit\}$
$B=\{3\clubsuit,\ 3\diamondsuit,\ 3\heartsuit,\ 3\spadesuit\}$



(ii) Complements

A: red cards in U
A': black cards in U

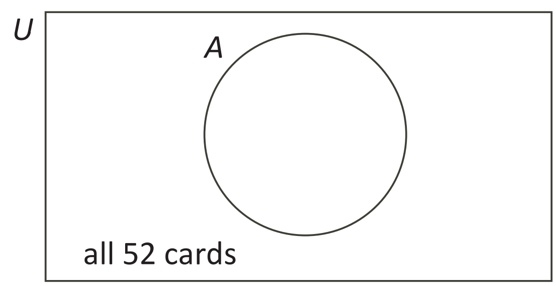
$A = \{$13 hearts and 13 diamonds$\}$
$A' = \{$13 clubs and 13 spades$\}$



(ii) Empty subset

A: cards with the number 15 on them in U

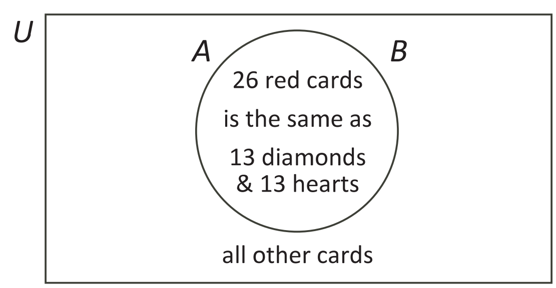
$A = \{\ \}$



(iv) Equal Sets

A: red cards in U
B: hearts and diamonds in U

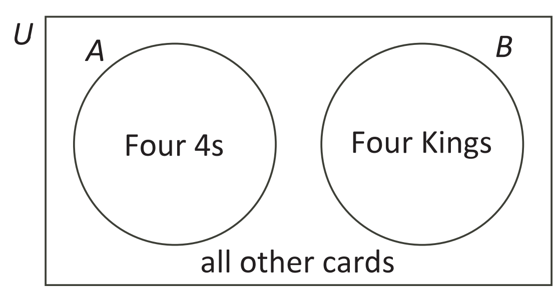
$A = \{$26 red cards$\}$
$B = \{$13 hearts and 13 diamonds$\}$



(v) Equivalent Sets

A: the fours in U
B: the Kings in U

$A = \{4\clubsuit\, 4\diamondsuit,\ 4\heartsuit,\ 4\spadesuit\}$
$B = \{K\clubsuit\, K\diamondsuit,\ K\heartsuit,\ K\spadesuit\}$



And Still More Terminology

For two sets A and B, the **union of** A **and** B is the set, denoted $A \cup B$, containing all the elements of the universal set which are either in set A or in set B or in both sets.

From a previous example,

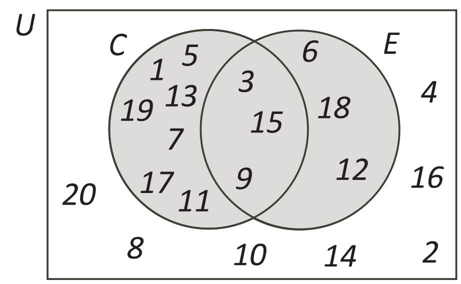
$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

C : the odd integers in U $\quad C = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

E : multiples of 3 in U $\quad E = \{3, 6, 9, 12, 15, 18\}$

Illustrate $C \cup E$ on a Venn diagram, list $C \cup E$ in set notation and describe $C \cup E$ in words.

The shaded region of the Venn diagram to the right shows $C \cup E$.



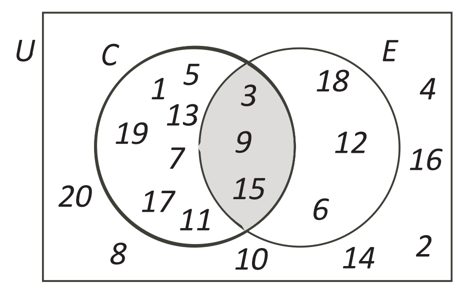
$C \cup E = \{1, 3, 5, 6, 7, 9, 11, 12, 13, 15, 17, 18, 19\}$

We can describe the union in words as follows: "$C \cup E$ is the set containing all odd integers less than or equal to 20 and all even multiples of 3 less than or equal to 20."

For two sets A and B, the **intersection of** A **and** B is the set, denoted $A \cap B$, containing all the elements of the universal set which are both in set A and in set B.

Illustrate $C \cap E$ on a Venn diagram, list $C \cap E$ in set notation and describe $C \cap E$ in words.

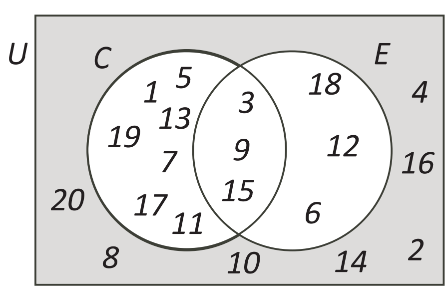
The shaded region of the Venn diagram to the right shows $C \cap E$.



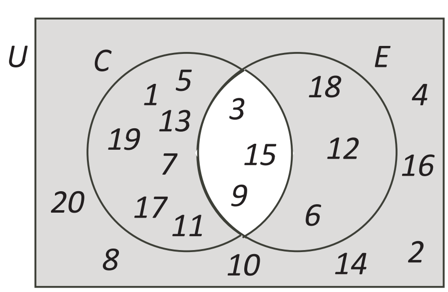
$C \cap E = \{3, 9, 15\}$

We can describe the intersection in words as follows: "$C \cap E$ is the set containing all odd integers less than or equal to 20 that are multiples of 3."

Illustrate $(C \cup E)'$ and $(C \cap E)'$on a Venn diagram. List each in set notation.



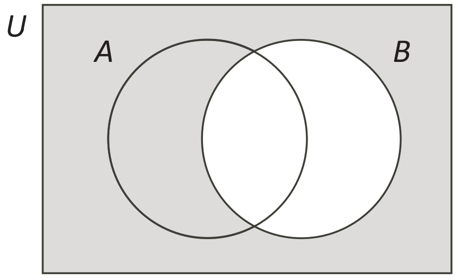
$(C \cup E)' = \{2, 4, 8, 10, 14, 16, 20\}$

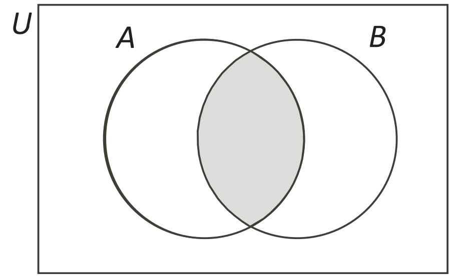


$(C \cap E)' = \{1, 2, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20\}$

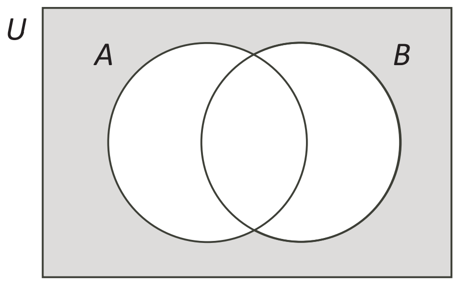
Example 9

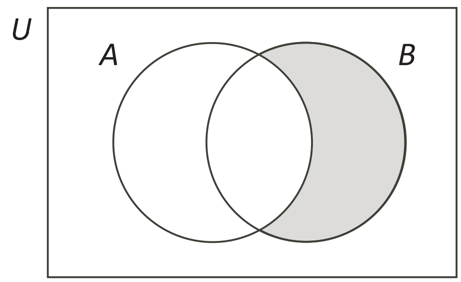
The shaded area of each Venn diagram represents some sort of intersection or union or complement or combination of sets A and B. Write an expression that represents the Venn diagram shown.

a) 

(b) 

The unshaded section in part (a) is B. The shaded section is everything but B. Therefore the shaded section is B'. In part (b), the shaded section shows the elements in both A and B and is therefore the intersection $A \cap B$.

c) 

(d) 

In part (c) the unshaded section contains elements in A, elements in B and elements in both A and B and is therefore $A \cup B$. Therefore the shaded region is the complement $(A \cup B)'$. In part (d), the shaded region contains elements in B only. We need to make the notation totally eliminate A. A' does that. If we then find the intersection of A' and B, we will have the shaded region. Therefore the answer is $A' \cap B$.

Counting Subsets

For disjoint subsets, A and B, $n(A \cup B) = n(A) + n(B)$. This result basically says that there is no intersection between the two subsets and to find the number in the union, we determine the number in each disjoint subset and add the result together.

Example 10

Given the universal set U, and subsets A and B described below, determine $n(A \cap B)$ and $n(A \cup B)$.

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

A : the numbers in U with 2 as one of the digits

B : multiples of 7 in U

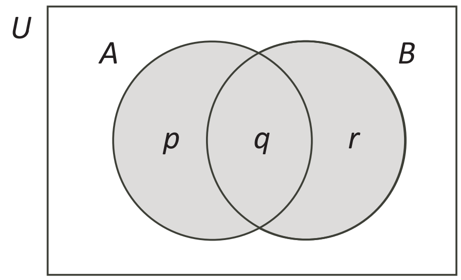
$A = \{2, 12, 20\}$ and $n(A) = 3$. $B = \{7, 14\}$ and $n(B) = 2$. Since A and B have no elements in common, $A \cap B = \varnothing$ and $n(A \cap B) = 0$. Therefore A and B are disjoint.

$n(A \cup B) = n(A) + n(B) = 3 + 2 = 5$.

Counting Subsets (continued)

For subsets, A and B, where $A \cap B \neq \varnothing$, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

We will illustrate this result with a Venn diagram.

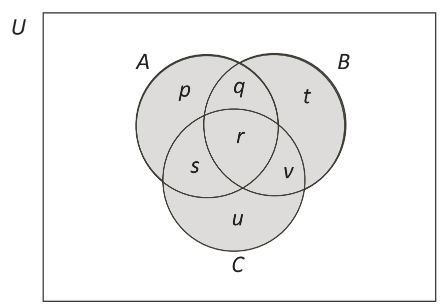


Let $n(A \cap B) = q$, $n(A) = p + q$ and $n(B) = q + r$.

$$\begin{aligned} n(A) + n(B) - n(A \cap B) &= (p+q) + (q+r) - (q) \\ &= p + q + q + r - q \\ &= p + q + r \\ &= n(A \cup B) \end{aligned}$$

For subsets, A, B and C,
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

We will illustrate this result with a Venn diagram.



Let $n(A \cap B \cap C) = r$, $n(A \cap B) = q + r$, $n(A \cap C) = s + r$, $n(B \cap C) = r + v$, $n(A) = p + q + r + s$, $n(B) = q + r + t + v$, and $n(C) = r + s + u + v$.

$$\begin{aligned} & n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) \\ = \ & (p+q+r+s) + (q+r+t+v) + (r+s+u+v) - (q+r) - (s+r) - (r+v) + (r) \\ = \ & p+q+r+s+q+r+t+v+r+s+u+v-q-r-s-r-r-v+r \\ = \ & p + 2q + 4r + 2s + t + u + 2v - q - 3r - s - v \\ = \ & p+q+r+s+t+u+v \\ = \ & n(A \cup B \cup C) \end{aligned}$$

Example 11

In a certain village, almost all of the people living there are farmers, 21 of them cultivate cassava and 18 cultivate groundnut. If 9 villagers cultivate both cassava and groundnut, and there are 36 people in the village, how many are not farmers?

Let $U = \{$the people who live in the village$\}$.
Let $C = \{$the people who live in the village who farm cassava$\}$.
Let $G = \{$the people who live in the village who farm groundnut$\}$.

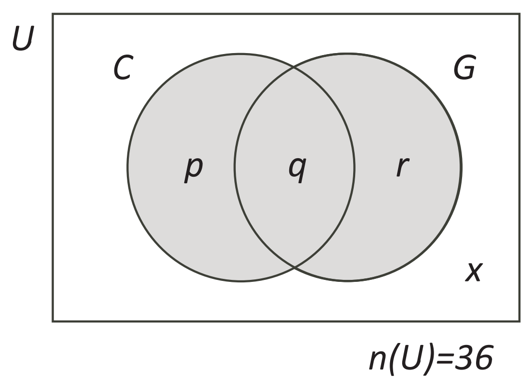
We want the complement of the union of C and G, $n((C \cup G)')$.

Then $n(U) = 36$, $n(C) = 21$, $n(G) = 18$ and $n(C \cap G) = 9$

$$\begin{aligned} n(C \cup G) &= n(C) + n(G) - n(C \cap G) \\ &= 21 + 18 - 9 \\ &= 30 \\ \text{We know that} \quad n(U) &= n(C \cup G) + n((C \cup G)') \\ 36 &= 30 + n((C \cup G)') \\ \therefore n((C \cup G)') &= 6 \end{aligned}$$

In the village, there are 6 people who are not farmers.

We can also solve the problem by using a Venn diagram



$q = 9$

$p + q = 21$ so $p = 12$

$q + r = 18$ so $r = 9$

$x + p + q + r = n(U)$

$x + 12 + 9 + 9 = 36$

$x + 30 = 36$

$x = 6$

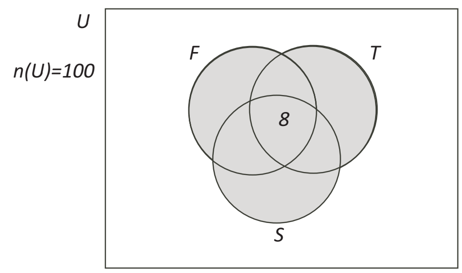
Example 12

A school sport's team consists of 100 students. 52 students do field events, 38 students do track events, 41 students swim, 17 students do track and swimming, 22 students do field and track events, 15 students do field events and swimming, and 8 students do all three.

a) Draw a Venn diagram illustrating the data.
b) How many students do only track events?
c) How many students do field events and swimming but not track events?
d) How many students are involved in exactly one of the three events?
e) How many students participate in field events or track events or swimming?

Let $U = \{$the students on the sport's team$\}$.
Let $F = \{$the students on the sport's team in Field events$\}$.
Let $T = \{$the students on the sport's team in Track events$\}$.
Let $S = \{$the students on the sport's team who swim$\}$.

We know that $n(F \cap T \cap S) = 8$. This is shown in the first Venn diagram.



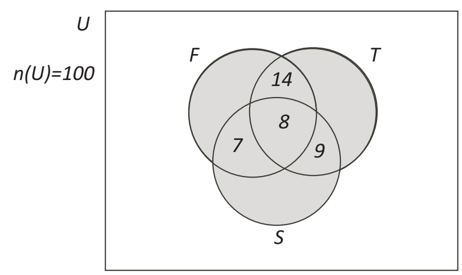
From this we can determine the three other numbers which make up the intersections $F \cap T$, $F \cap S$, $S \cap T$.

$n(F \cap T) = 22$ so the missing number in $F \cap T$ is $22 - 8 = 14$.

$n(F \cap S) = 15$ so the missing number in $F \cap S$ is $15 - 8 = 7$.

$n(S \cap T) = 17$ so the missing number in $S \cap T$ is $17 - 8 = 9$.

This is shown in the second Venn diagram.



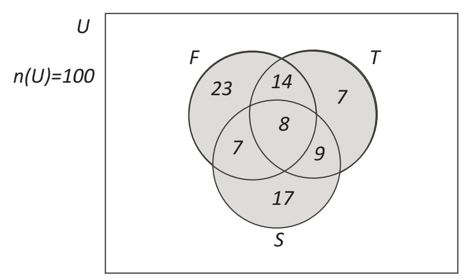
From this we can determine the three other numbers which make up the the individual sets F, S, T.

$n(F) = 52$ so the missing number in F is $52 - 14 - 8 - 7 = 23$.

$n(S) = 41$ so the missing number in S is $41 - 7 - 8 - 9 = 17$.

$n(T) = 38$ so the missing number in T is $38 - 14 - 8 - 9 = 7$.

This is shown in the third Venn diagram.



Finally, we can determine the number not in $F \cup T \cup S$ by subtracting the number of students in the union from 100, the total number of students on the sport's team.
$\therefore n((f \cup T \cup S)') = 100 - 23 - 14 - 7 - 7 - 8 - 9 - 17 = 100 - 85 = 15.$

At this point we can "easily" answer each of the parts.

(b) The number of students doing only track is 7.

(c) 7 students do field and swimming but not track.

(d) $23 + 7 + 17 = 47$ students are involved in exactly one of the three events.

(e) The number of students who do at least one of Field, Track or Swimming is 85.