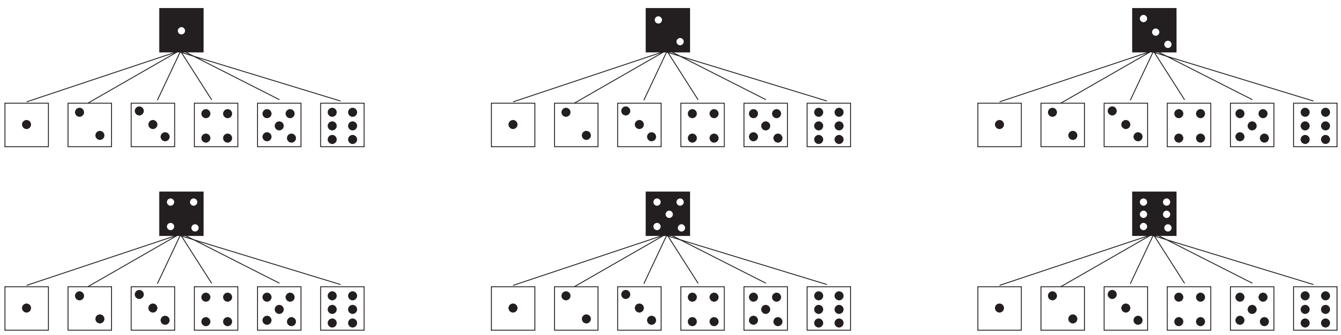
Mathematics Teachers Enrichment Program
MTEP 2012
Probability

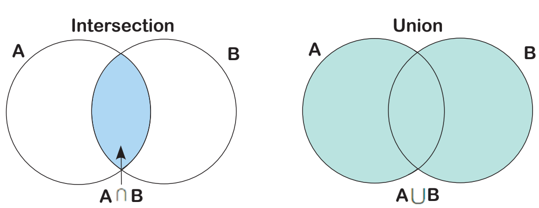
Rolling a pair of dice is an example of an **experiment**. When each die is rolled there are 6 **equally likely outcomes**, or there is an equal chance that each of the numbers 1,2,3,4,5,6 is going to land face up. The set of *all* possible outcomes of an experiment is called the **sample space** of the experiment. So, for the experiment of rolling a pair of dice, say one white and one black, the sample space consists of 36 possible outcomes. The tree diagrams below illustrate the possible outcomes.



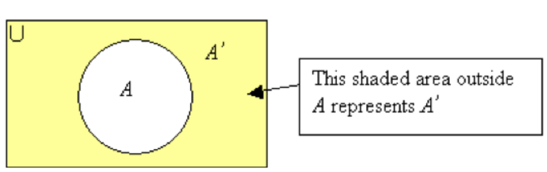
The diagrams show that for each possible roll on the black die, there are 6 possible rolls with the white die. The sample space could have also been written as a set of ordered pairs: {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)} where the first coordinate represents the black die and the second coordinate represents the white die.

A **trial** of the experiment is one attempt of the experiment. So for the experiment of rolling a pair of dice, one trial would be rolling the pair of dice once. An **event** is a set of possible outcomes or a **subspace** of the sample space. For example, in the experiment of rolling a pair of dice, the event of having the sum of the roll equal to 3 consists of 2 outcomes or {(1,2),(2,1)}

Consider two events A and B. Each of these events is a subset of the same sample space. The **intersection**, $A \cap B$, of two sets A and B is the set of all elements common to both A and B. The **union**, $A \cup B$, is the set of elements that belong to A or B or to both A and B.



Given a set U and a subset A of U, the **complement** of A, A', in U is the set of elements in U that do not belong to A. Note that $n(A) + n(A') = n(U)$.



If the sample space of an experiment consists of N equally likely outcomes, the probability of an event E of the sample space S is given by:

$$\begin{aligned}\text{Probability of event } E &= \frac{\text{number of elements in } E}{\text{number of elements in } S}\\ P(E) &= \frac{\text{n}(E)}{N}\end{aligned}$$

For example, in the experiment of rolling a single die, each face of the die has an equal chance of landing face up. So the probability of rolling a 6 is:

$$P(\text{Rolling a 6}) = \frac{\text{Number of 6's on die}}{\text{Number of faces on die}} = \frac{1}{6}$$

When an event is certain to happen, its probability is 1. When an event is certain *not* to happen, its probability is 0. All probabilities must equal 0 or 1 or lie between them. So, we can write for any probability p, $0 \leq p \leq 1$.

Given that the probability that an event E will occur is p, then the probability that the event E will *not* occur is $q = 1 - p$.

In the experiment of rolling a die, the probability of *not* rolling a 6 is equal to one minus the probability of rolling a 6, which we calculated above.

$$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Two events, E_1 and E_2, are said to be **mutually exclusive** if they cannot both occur simultaneously. For example, in the experiment of rolling a pair of dice, two events that are considered mutually exclusive are the event of having the dice sum to 3 and the event of having the dice sum even. Then the probability that two mutually exclusive events E_1 or E_2 will occur is equal to the sum of their probabilities.

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

If the two events E_1 and E_2 are *not* mutually exclusive, then the probability that event E_1 or E_2 will occur is given by the **sum rule**:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Two events, E_1 and E_2, are said to be **independent** when neither event has an influence on the other. For example, in the experiment of rolling a pair of dice, each outcome of a trial is an independent event since the previous roll has no influence on the outcome of the next roll. Then the probability that two independent events will occur is equal to the product of the probabilities of E_1 and E_2.

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2)$$

In other words, if the first of two tasks can be done in a ways and, for each of those ways, the second task can be done in b ways, then by the **product rule**, together the two tasks can be done in $a \times b$ ways.

Examples:

1. A card is drawn from a deck of cards. What is the probability that it is:
   a) a 4? b) a face card? c) a spade?

   Solution:

   There are 52 cards in a deck and each card has the same chance of being drawn.

   a) There are four 4's in a deck, so the probability is:

   $$P(\text{drawing a } 4) = \frac{\text{Number of 4's in the deck}}{\text{Number of cards in the deck}} = \frac{4}{52} = \frac{1}{13}$$

   b) There are 3 face cards per suit and 4 suits in a deck, thus, there are 12 face cards in the deck. The probability of drawing a face card is:

   $$P(\text{drawing a face card}) = \frac{\text{Number of face cards in the deck}}{\text{Number of cards in the deck}} = \frac{12}{52} = \frac{3}{13}$$

   c) There are 13 spades in the deck. The probability of drawing a spade is:

   $$P(\text{drawing a spade}) = \frac{\text{Number of spades in the deck}}{\text{Number of cards in the deck}} = \frac{13}{52} = \frac{1}{4}$$

2. A number is chosen at random from the integers 5 to 25 inclusive.
   a) What is the sample space?
   b) Find the probability that the number is a multiple of 5 or 7.
   c) Find the probability that the number is a multiple of 5 or 3.

Solution:

a) The sample space is: $\{5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25\}$. There are a total of 21 numbers in the sample space. This number can be calculated by subtracting the smallest number from the largest number and adding 1. So the number of numbers in the sample space is $25 - 5 + 1 = 21$, as before.

b) From the sample space, there are 5 numbers that are a multiple of 5, $\{5,10,15,20,25\}$. The probability of choosing a number that is a multiple of 5 is

$$P(\text{multiple of } 5) = \frac{\text{Number of multiples of } 5}{\text{Number of integers in sample space}} = \frac{5}{21}$$

There are 3 integers in the sample space that are a multiple of 7, $\{7,14,21\}$. The probability of choosing a number that is a multiple of 7 is:

$$P(\text{multiple of } 7) = \frac{\text{Number of multiples of } 3}{\text{Number of integers in sample space}} = \frac{3}{21} = \frac{1}{7}$$

Since the events are mutually exclusive (no elements in common), the probability of choosing a multiple of 5 or 7 is:

$$P((\text{multiple of } 5) \cup (\text{multiple of } 7)) = P((\text{multiple of } 5) + P(\text{multiple of } 7)) = \frac{5}{21} + \frac{1}{7} = \frac{8}{21}$$

c) From the sample space, there are 5 numbers that are a multiple of 5, $\{5,10,15,20,25\}$ and 7 multiples of 3, $\{6,9,12,15,18,21,24\}$. These events are not mutually exclusive since the element 15 is both a multiple of 3 and 5.

$$P((\text{multiple of } 5) \cup (\text{multiple of } 3)) = P(\text{multiple of } 5) + P(\text{multiple of } 3) - P(\text{multiple of 5 and 3})$$

$$\therefore P((\text{multiple of } 5) \cup (\text{multiple of } 3)) = \frac{5}{21} + \frac{7}{21} - \frac{1}{21} = \frac{11}{21}$$

3. In a game, a fair die is rolled once and two unbiased coins are tossed once. What is the probability of obtaining 3 and at least one tail.

Solution:

The two events are independent since the outcome of rolling the die does not affect the outcome of tossing the two coins.

$$P(\text{Rolling a 3}) = \frac{\text{Number of threes on die}}{\text{Number of possible rolls}} = \frac{1}{6}$$

The sample space for the experiment of tossing two coins is: {HH,HT,TH,TT}. Three of the events contain at least one tail.

$$P(\text{At least one tail}) = \frac{\text{Number of trials with at least one tail}}{\text{Number of trials}} = \frac{3}{4}$$

Then, the probability of obtaining a 3 and at least one tail is:

$$\begin{aligned} P((\text{Rolling a 3}) \cup (\text{At least one tail})) &= P(\text{Rolling a 3}) \times P(\text{at least one tail}) \\ &= \frac{1}{6} \times \frac{3}{4} \\ &= \frac{3}{24} \\ &= \frac{1}{8} \end{aligned}$$

4. A bag contains 5 black marbles and 5 red marbles. A marble is taken from the bag and then a second marble is taken from the bag. Find the probability of drawing 2 red marbles if the first marble is:
a) replaced before drawing the second   b) not replaced

Solution:

a) First draw: There are 10 marbles of which 5 are red. So the probability of drawing a red marble is:

$$P(\text{red}) = \frac{5}{10} = \frac{1}{2}$$

Second draw: Since the first red marble was replaced, the probability of drawing a second red marble is still $\frac{1}{2}$. Then the probability of drawing 2 red marbles in succession is:

$$P(\text{2 red marbles in succession with replacement}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

b) From part a), we know that the probability of choosing a red marble on the first draw is $\frac{1}{2}$.
Second draw: Since the first red marble that was drawn was not replaced, there are only 9 marbles of which 4 are red. The probability of drawing a red marble on the second draw is:

$$P(\text{2nd red marble}) = \frac{4}{9}$$

Then the probability of drawing two red marbles in succession without replacing the first marble is:

$$P(\text{2 red marbles in succession without replacement}) = \frac{1}{2} \times \frac{4}{9} = \frac{4}{18} = \frac{2}{9}$$

Exercises:

1. List the sample space for each of the following experiments:
   a) A card is drawn from a deck of playing cards and you are interested in the colour of the suit.
   b) A card is drawn from a deck of playing cards and you are interested in the suit of the card.
   c) A coin is tossed once and a fair die is rolled once.

2. In the experiment of rolling a fair die and recording the number, list the possible outcomes for each of the following events:
   a) The number is odd.
   b) The number is prime.
   c) The number is less than 5.
   d) The number is more than 7.

3. A fair die is rolled once. What is the probability of rolling a:
   a) 2 or 5?
   b) 3 or 4?
   c) Neither 6 or 1?

4. A card is drawn from a deck. What is the probability that it is:
   a) an ace?
   b) a black card?
   c) a red face card?
   d) a club face card?
   e) The 7♣?

5. A man has 15 marbles in a bag. Six of them are black, 5 are blue and the rest are red.
   a) If a marble is drawn at random, what is the probability that it is:
      i) Not black
      ii) Not red
   b) If two marbles are drawn at random, one after the other, what is the probability that both of them will be:
      i) blue, if there is no replacement
      ii) red, if there is a replacement

6. The following data was collected from a mathematics class.

Age	**Girls**	**Boys**
13	3	2
14	5	6
15	8	9
16	1	3

   If a student is selected from the class at random what is the probability that the student is:
   a) a boy?
   b) 15 years old?
   c) a 13 year old girl?
   d) a 15 year old boy?
   e) less than 15 years old?
   f) 17 years old?

7. A fair die is rolled three times. What is the probability that the roll is 5 or greater each time?

8. A box contains 10 marbles, 7 of which are black and 3 are red. Two marbles are drawn, one after the other, without replacement. Find the probability of getting:
   a) A red, then a black
   b) Two black marbles.

9. The letters in the phrase “GO MTEP” are written on separate slips of paper and placed in a bag. What is the probability that two slips drawn simultaneously will both show:
   a) vowels?
   b) consonants?

10. A multiple choice test has 4 questions. Each question has 5 responses, only one of which is correct. If Robin attempts this test by guessing, what is the probability that she will get all 4 questions right?

11. From the 2006 Examiners' Report
A man P has 5 red, 3 blue, and 2 white buses. Another man Q has 3 red, 2 blue, and 4 white buses. A bus owned by P is involved in an accident with a bus belonging to Q. Calculate the probability that the two buses are **not** of the same colour.

12. Two dice are thrown, one red and one blue. What is the probability that the number shown on one die is a multiple of the number on the other?
Note: A multiple of a number is the product of that number multiplied by a whole number. For example, 2 is a multiple of 1 since $1 \times 2 = 2$.

13. There are twelve cards numbered 1 to 12. A card is selected at random. What is the probability that:
    a) the card is either even or a perfect square?
    b) the card is even and a perfect square?

14. A bag contains two red marbles and three blue marbles. A second bag contains three red marbles and two blue marbles. A marble is taken from each bag.
    a) Make a diagram to represent all the possible outcomes.
    b) What is the probability that:
        i) both marbles are red
        ii) both marbles are blue?
    c) Find the probability that one marble is red and the other is blue.

15. A blue die and a red die are rolled simultaneously. What is the probability of obtaining:
    a) a total score of 7?
    b) a total score of 10?
    c) a total score of 7 or 10?
    d) a total score not greater than 10?

16. From the 2006 Examiners' Report
    a) Two pupils are chosen at random from a group of 4 boys and 5 girls. Find the probability that the two pupils chosen would be boys.
    b) Twenty percent of the total production of transistors produced by a machine are below standard. If a random sample of 6 transistors produced by the machine is taken, what is the probability of getting:
        i) exactly 2 standard transistors?
        ii) exactly 1 standard transistor?
        iii) at least 2 standard transistors?
        iv) at most two standard transistors?

17. A blue die and a red die are rolled simultaneously. A two-digit number is formed with the number on the blue die giving the tens digit and the number on the red die giving the units digit. For example, a 3 on the blue die and 5 on the red die gives the two-digit number 35.
    a) Draw all the possible outcomes.
    b) What is the probability of obtaining a two-digit number which is:
        i) greater than 30
        ii) exactly divisible by 11
        iii) prime?
    c) What is the probability of obtaining a two-digit number that is either a perfect square or exactly divisible by 7?

18. A box contains 5 red marbles and 3 green marbles. Two marbles are drawn one after the other without replacement. Find the probability of drawing first a red, then a green marble.