Linear Transformations

A **transformation** of an object is a change in position or dimension (or both) of the object. The resulting object after the transformation is called the **image**.

If the image of the object has the same dimensions as the object, the transformation is called a **congruency**. Two shapes are said to be congruent when their interior angles are the same.

Types of Transformations:

Translation:

A **translation** is a transformation of an object in which each point moves the same distance and in the same direction. Translations may occur in any direction.

Properties of Translations:

- A translation preserves length and slope of a line.
- A translation by a vector of the form $\begin{bmatrix} h \\ k \end{bmatrix}$ shifts the x-coordinate of each point by h units, and the y-coordinate of each point by k units.

Reflection:

A **reflection** is a transformation of an object that produces a mirror image of the object with respect to a line. The line about which the object is reflected is called the **axis of symmetry**.

Properties of Reflections:

- A reflection preserves length but does not preserve slope or orientation of a line.
- A reflection in the line $y = 0$ (x-axis) changes the point (x, y) to $(x, -y)$
- A reflection in the line $x = 0$ (y-axis) changes the point (x, y) to $(-x, y)$
- A reflection in the line $y = x$ changes the point (x, y) to (y, x)
- A reflection in the line $y = -x$ changes the point (x, y) to $(-y, -x)$

Rotation:

A **rotation** is a transformation where the object is turned. The point about which the object is rotated is called the **center of rotation**.

Properties of Rotations:

- A rotation preserves length but does not necessarily preserve slope of a line.
- A $90°$ rotation ($\frac{1}{4}$ turn) anticlockwise about the origin changes the point (x, y) to $(-y, x)$.
- A $180°$ rotation ($\frac{1}{2}$ turn) clockwise or anticlockwise about the origin changes the point (x, y) to $(-x, -y)$.
- A $270°$ rotation ($\frac{3}{4}$ turn) anticlockwise changes about the origin the point (x, y) to $(y, -x)$.

Dilatation

A **dilatation** is a transformation in which the object is magnified (made larger) or diminished (made smaller) by a factor of k. The factor k is called the **scale factor**.

Properties of Dilatations:

- A dilatation preserves slope but does not preserve length or area (except when $\mid k \mid = 1$)
- If $0 < k < 1$, the dilatation is a reduction
- If $k > 1$, the dilatation is an enlargement
- if $k < 0$, the object is also reflected across the x axis.

Composition of Linear Transformations:

When a question requires multiple linear transformations to be performed, perform each linear transformation one at a time to find the image points, or line, after each transformation. Then the image points can be used to perform the next linear transformation.

Finding a Matrix of a Linear Transformation:

To find the matrix of a linear transformation given two points, (x_1, y_1) and (x_2, y_2), and the image points after the points have been transformed, (x_1', y_1') and (x_2', y_2'):

1. Let the transformation matrix be represented by: T= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
2. Multiply each point by the transformation matrix T and set it equal to its image point. This will create four equations of the form:
   $ax_1 + by_1 = x_2'$
   $cx_1 + dy_1 = y_2'$
   $ax_2 + by_2 = x_2'$
   $cx_2 + dy_2 = y_2'$
3. Then, using the four equations, use substitution or elimination to solve for a, b, c, and d.

Once the transformation matrix is found, we can determine the image point of any point on a line by multiplying the point by the transformation matrix. (ie. $[T]\vec{x}$)

Note: Matrix multiplication is *not* cumulative. This means that $XT \neq TX$

To multiply a 2×2 matrix (a matrix with 2 rows and 2 columns) by a 2×1 matrix (a matrix with 2 rows and 1 column):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Linear Transformation of a Line:

To find the equation of the image of a line under a translation, reflection, rotation, or dilatation:

1. Find the coordinates of any two points on the line.
2. Find the coordinates of the images of the two points from step 1.
3. Use the two image points from step 2 to find the slope of the image line.
4. Use one of the image points, the slope from step 3, and the equation of a line to find the equation of the image line.

Examples:

1. a) Given that A(1,2), B(-2,-1) and C(-6,2m) are three points in the xy plane and that m is a constant, find: i) $\overrightarrow{BA}$ ii) The value of m if $\overrightarrow{BC} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$

b) A(3,-6), B(6,-2), and C(x,y) are three points in the xy plane. If $\frac{1}{3}\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{AB}$, find the coordinates of C.

Solution:

a) i)
$$\begin{aligned} \overrightarrow{BA} &= A - B \\ &= (1,2) - (-2,-1) \\ &= (1-(-2), 2-(-1)) \\ &= (3,3) \end{aligned}$$

ii)
$$\begin{aligned} \overrightarrow{BC} &= C - B \\ (-4,5) &= (-6,2m) - (-2,-1) \\ (-4,5) &= (-6-(-2), 2m-(-1)) \\ (-4,5) &= (-4, 2m+1) \\ 5 &= 2m+1 \quad \text{Equate the } y \text{ components to solve for m} \\ 4 &= 2m \\ m &= 2 \end{aligned}$$

b)
$$\begin{aligned} \frac{1}{3}\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} &= \overrightarrow{AB} \\ \frac{1}{3}(A-O) + (B-O) + (C-O) &= B - A \\ \frac{1}{3}[(3,-6)-(0,0)] + [(6,-2)-(0,0)] + [(x,y)-(0,0)] &= (6,-2)-(3,-6) \\ \frac{1}{3}(3,-6) + (6,-2) + (x,y) &= (6-3, -2-(-6)) \\ (1,-2) + (6,-2) + (x,y) &= (3,4) \\ (7,-4) + (x,y) &= (3,4) \\ (x,y) &= (3,4) - (7,-4) \\ (x,y) &= (-4,8) \end{aligned}$$

2. a) Plot the triangle ABC with vertices A(1,1), B(4,1), and C(2,3)

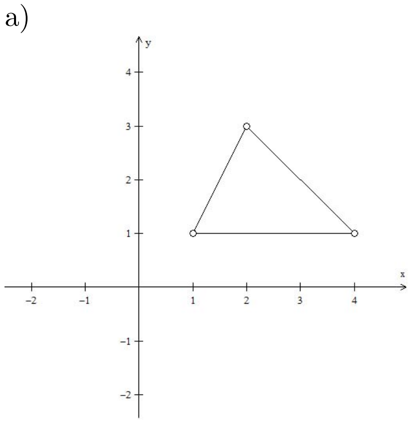
b) Draw the image $\triangle A_1B_1C_1$ of $\triangle ABC$ when it has been translated by the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

c) Draw the image $\triangle A_2B_2C_2$ of $\triangle ABC$ when it has been rotated anticlockwise 90° about the origin.

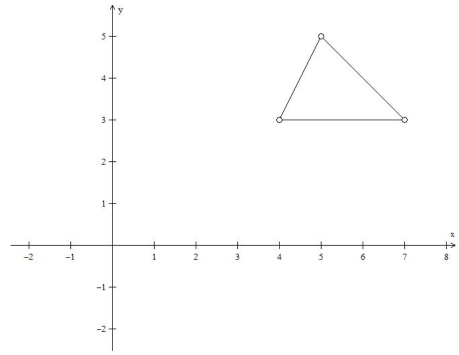
d) Draw the image $\triangle A_3B_3C_3$ of $\triangle ABC$ when it is reflected across the line $y = 0$.

e) Draw the image $\triangle A_4B_4C_4$ of $\triangle ABC$ when it has been enlarged by a scale factor of 2. **Solution:**

a)

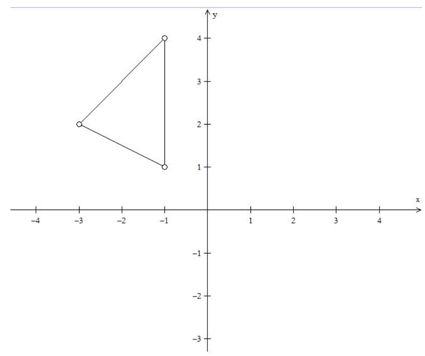


b)



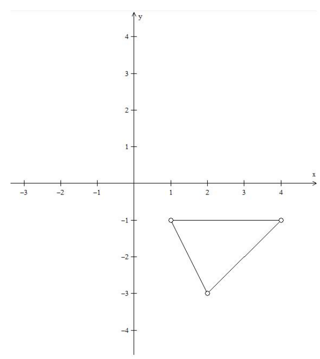
We can find A_1, B_1, and C_1 by translating the x coordinates 3 units and the y coordinates 2 units.
$A(1,1) \rightarrow A_1(1+3, 1+2) = (4,3)$
$B(4,1) \rightarrow B_1(4+3, 1+2) = (7,3)$
$C(2,3) \rightarrow C_1(2+3, 3+2) = (5,5)$

c)



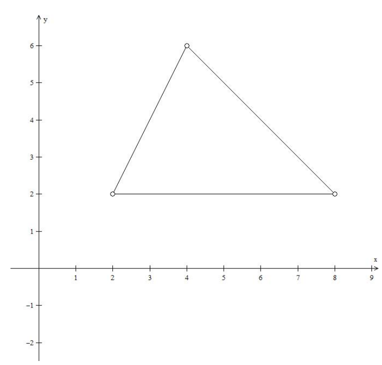
We can find A_1, B_1, and C_1 by changing each (x,y) to (-y,x).
$A(1,1) \rightarrow A_1(-1,1)$
$B(4,1) \rightarrow B_1(-1,4)$
$C(2,3) \rightarrow C_1(-3,2)$

d)



We can find A_1, B_1, and C_1 by negating the y coordinate of each pair.
$A(1,1) \rightarrow A_1(1,-1)$
$B(4,1) \rightarrow B_1(4,-1)$
$C(2,3) \rightarrow C_1(2,-3)$

e)

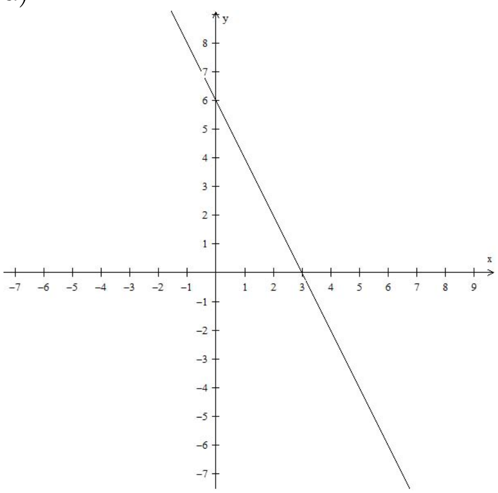


We can find A_1, B_1, and C_1 by multiplying both the x and y coordinates by the scale factor (2).
$A(1,1) \rightarrow A_1(2\times 1, 2\times 1) = (2,2)$
$B(4,1) \rightarrow B_1(2\times 4, 2\times 1) = (8,2)$
$C(2,3) \rightarrow C_1(2\times 2, 2\times 3) = (4,6)$

3. a) Graph the line $2x + y - 6 = 0$
b) Find the equation of the image of the line in part (a) under a $90°$ anticlockwise rotation about the origin.
c) Graph the image line on the same axis as part (a).

Solution:

a)



b)

To find the equation of the image line, we first start by finding two points that lie on the original line. Let's choose (0,6) and (-2,10). Then we find the image of these two points under the transformation (a 90° anticlockwise rotation).
The image point of (0,6) under the rotation is (-6,0) and the image point of (-2,10) under the rotation is (-10,-2).
We can now use the two image points to find the slope of our image line.

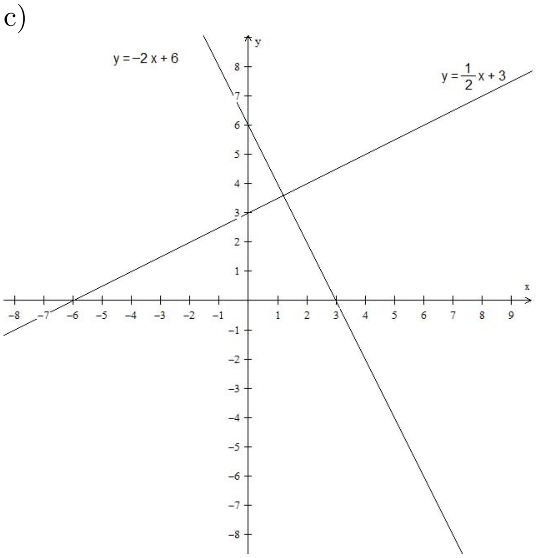
$$\begin{aligned} \text{Slope} &= \frac{\Delta y}{\Delta x} \\ &= \frac{0-(-2)}{-6-(-10)} \\ &= \frac{2}{4} \\ &= \frac{1}{2} \end{aligned}$$

Then, we can use the slope and one point to find the equation of the image line. Let's use (-6,0).
The equation of a line is $y = mx + b$ where m is slope and b is the y-intercept

$$\begin{aligned} y &= mx + b \\ y &= \frac{1}{2}x + b \qquad \text{plug in (-6,0) and solve for b} \\ 0 &= \frac{1}{2}(-6) + b \\ 0 &= -3 + b \\ b &= 3 \end{aligned}$$

So the equation of the image line is: $y = \frac{1}{2}x + 3$ or $y - \frac{1}{2}x - 3 = 0$

c)



4. A linear transformation T maps A(1,2) to A_1(-4,6) and B(6,3) to B_1(3,0). Find each of the following:
   a) The matrix of T.
   b) The image of the point K(-5,-2) under T.

   Solution:

   a) Let T=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then,

   $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$ gives us the following equations:

   $a + 2b = -4$ .... (1)
   $c + 2d = 6$ .... (2)
   and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ gives us:
   $6a + 3d = 3$ .... (3)
   $6c + 3d = 0$ .... (4)

   Substitute (1) into (3):

   $$\begin{aligned} 6(-4 - 2b) + 3b &= 3 \\ -24 - 12b + 3b &= 3 \\ -9b &= 27 \\ b &= -3 \end{aligned}$$

   Substitute b=-3 into (1)

   $$\begin{aligned} a + 2(-3) &= -4 \\ a - 6 &= -4 \\ a &= 2 \end{aligned}$$

   Substitute (2) into (4):

   $$\begin{aligned} 6(6 - 2d) + 3d &= 0 \\ 36 - 12d + 3d &= 0 \\ -9d &= -36 \\ d &= 4 \end{aligned}$$

   Substitute d=4 into (2)

   $$\begin{aligned} c + 2(4) &= 6 \\ c + 8 &= 6 \\ c &= -2 \end{aligned}$$

   Therefore, $T = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$

   b) $\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -5 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

Exercises:

1. Given that P(3,1), Q(6,-1), and R(-3,3m) are three points in the xy plane and that m is a constant, find:
   a) $\overrightarrow{PQ}$
   b) The value of m if $\overrightarrow{PR} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$
   c) $\overrightarrow{OP} + \overrightarrow{OQ} + \frac{1}{3}\overrightarrow{OR}$

2. Given that A(-2,4), B(5,-1), and C(x,y) are three points in the xy plane.
   If $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BC} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, find the coordinates of C.

3. In $\triangle DEF$, $\overrightarrow{DE} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\overrightarrow{FE} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$. Find $\overrightarrow{DF}$.

4. a) Plot the triangle RST with vertices R(-1,0), S(-3,2), and T(-4,-3)
   b) Draw the image $\triangle R_1S_1T_1$ of $\triangle RST$ when it has been translated by the vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$
   c) Draw the image $\triangle R_2S_2T_2$ of $\triangle RST$ when it has been rotated clockwise 90° about the origin.
   d) Draw the image $\triangle R_3S_3T_3$ of $\triangle RST$ when it is reflected across the line $x = 2$.
   e) Draw the image $\triangle R_4S_4T_4$ of $\triangle RST$ when it has been enlarged by a scale factor of $\frac{1}{2}$ about the origin.

5. a) Graph the line $2x + y - 6 = 0$.
   b) Determine the equation of the image of the line from part (a) under a 90° anticlockwise rotation.
   c) Graph the image line on the same axis as part (a).

6. Draw each of the following on the same graph, clearly indicating the coordinates of each vertex.
   a) A quadrilateral $ABCD$ has vertices A(2,2), B(6,2), C(8,8), and D(4,8).
   b) The image $A_1B_1C_1D_1$ of quadrilateral $ABCD$ under a reflection in the line $y = -x$.
   c) Find the equation of the line ΔD_1

7. From Examiners' Report 2006.
   a) Given that $A = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$ and $X = (3, 4)$, evaluate XA.
   b) A linear transformation T maps A(3,4) to A_1(-6,8) and B(5,2) to B_1(-10,4). Find each of the following:
   i) The matrix of T.
   ii) The image of the point K(1,-1) under T.

8. $\triangle PQR$ has coordinates P(-3,1), Q(-2,4), and R(0,4). If M is a reflection in the line x=1, R is a clockwise rotation of 90° about the point (2,0) and E is an enlargement by a scale factor of $\frac{-1}{2}$ about the origin. Find the image of $\triangle PQR$ after:
   a) M
   b) R
   c) M followed by R
   d) R followed by M
   e) E

9. Determine the equation of the line $3x - 5y - 15 = 0$ under the following linear transformations:
   a) A translation by the vector $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.
   b) Reflection in the line $y = x$.
   c) A 270° anticlockwise rotation.

10. $A_1(5,5), B_1(-5,10), C_1(0,20)$ are the images of $A(2,2), B(-2,4)$, and $C(0,8)$ after a transformation F.
    a) Draw the triangles ABC and $A_1B_1C_1$ on the same graph.
    b) Describe fully the transformation F.
    c) Find the coordinates of the image of $\triangle ABC$ after a rotation of 270° clockwise about the point (3,2)

11. From Examiners' Report 2006.
    a) Using a scale of 2cm to 2 units on both axes draw, on graph paper, two perpendicular axes $0x$ and $0y$ for $-8 \leq x \leq 8$ and $-8 \leq y \leq 8$.
    b) Draw the $\triangle PQR$ with vertices P(2,5), Q(2,-1), and R(8,4).
    c) Describe fully **two different** transformations that map P to Q.
    d) Draw the image $\triangle P_1Q_1R_1$ of $\triangle PQR$ under a 90° anticlockwise rotation about the origin.
    e) Find the vector $\overrightarrow{PP_1}$

12. $WXYZ$ is a quadrilateral with vertices W(1,1), X(2,-1), Y(4,0), and Z(3,1).
    a) Draw $WXYZ$ and clearly label each of its vertices.
    b) Draw the image $W_1X_1Y_1Z_1$ of $WXYZ$ under a reflection in the line y=2. Clearly label the coordinates of each vertex.
    c) S is a transformation which maps $WXYZ$ onto quadrilateral $W_2X_2Y_2Z_2$ with vertices W_2(-5,-3), X_2(-7,-3), Y_2(-8,-2), and Z_2(-6,-1). Draw $W_2X_2Y_2Z_2$ and describe the transformation S fully.

13. a) On the same axes, graph the two lines L_1 and L_2 where $L_1 : 2x-5y+10 = 0$ and $L_2 : 2x-5y-10 = 0$
    b) Describe a possible transformation in which L_1 is the image of L_2 assuming that the transformation is a:
    i) translation ii) reflection iii) rotation

14. a) Draw the $\triangle PQR$ with vertices P(1,2), Q(3,4), R(6,1).
    b) Draw the image:
    i) $\triangle P_1Q_1R_1$ of $\triangle PQR$ under a reflection in the x-axis
    ii) $\triangle P_2Q_2R_2$ of $\triangle P_1Q_1R_1$ under a reflection in the y-axis
    iii) $\triangle P_3Q_3R_3$ of $\triangle PQR$ under a 90° anticlockwise rotation about the origin
    c) Describe fully, the single transformation that takes:
    i) $\triangle PQR$ to $\triangle P_2Q_2R_2$
    ii) $\triangle P_2Q_2R_2$ to $\triangle P_3Q_3R_3$

15. a) Graph the line $y = \frac{3}{2}x - 3$
    b) On the same axes, graph the line in part (a) after it has been reflected in the line $y = x$.
    c) Determine the equation of the line in part b.
    d) Determine the matrix of the linear transformation.